Comparative Study of the Convergence Rates of Two Numerical Techniques

Ajay Kumar*
Old Dominion University, Norfolk, Va.
and
R. A. Graves Jr.†
NASA Langley Research Center, Hampton, Va.

Nomenclature

h	= nondimensional specific enthalpy, $\bar{h}/\bar{V}_\infty^2$
H	= nondimensional total enthalpy, $\bar{H}/\bar{V}_\infty^2$
M_∞	= freestream Mach number
n	= normal coordinate, $\bar{n}/\bar{R}_n$
p	= nondimensional pressure, $\bar{p}/\bar{\rho}_\infty \bar{V}_\infty^2$
r	= radius of body cross section, $\bar{r}/\bar{R}_n$
R	= time step factor in Eq. (5)
Re	= freestream Reynolds number, $\bar{\rho}_\infty \bar{V}_\infty \bar{R}_n/\bar{\mu}_\infty$
$\bar{R}_n$	= nose radius, m
s	= tangential coordinate, $\bar{s}/\bar{R}_n$
t	= nondimensional time, $\bar{t}\bar{V}_\infty/\bar{R}_n$
$\bar{T}_\infty$	= freestream temperature, K
u	= tangential velocity component, $\bar{u}/\bar{V}_\infty$
v	= normal velocity component, $\bar{v}/\bar{V}_\infty$
$\bar{V}_\infty$	= freestream velocity, m/s
α	= adjustable parameter in Eq. (7)
β	= $r+n\cos\theta$
β'	= stretching factor
δ	= shock standoff distance, $\bar{\delta}/\bar{R}_n$
θ	= body angle measured from the body axis
λ	= $1+n\kappa$
κ	= local curvature, $\bar{\kappa}\bar{R}_n$
ρ	= nondimensional density, $\bar{\rho}/\bar{\rho}_\infty$
$\bar{\rho}_\infty$	= freestream density, kg/m^3
μ	= nondimensional viscosity, $\bar{\mu}/\bar{\mu}_\infty$
$\bar{\mu}_\infty$	= freestream viscosity N s/m^2
σ	= Prandtl number
$(\Delta t)_{\rm CFL}$	= time increment permitted by the CFL condition

Superscripts

k	= time-step
$(\bar{\ })$	= dimensional quantities

Introduction

WITH the development of large-scale vector processing computers, a great deal of emphasis is being placed on developing more efficient, explicit methods which are highly suitable for such computers. These methods integrate the unsteady equations forward in time until a steady-state solution is obtained. A comparative study of several popular explicit finite-difference methods is made in Ref. 1 to determine their computational efficiencies using a simple model problem. This study is further extended in Ref. 2 by including a three-step method due to Stetter.[3] It is found in Ref. 2 that Stetter's method required a substantially fewer number of iterative steps, as compared to other schemes, to reach the steady-state solution. The purpose of this study is to further investigate the applicability of Stetter's method to a more complex problem of the hypersonic viscous flow over a blunt axisymmetric body used for planetary entry probes at zero angle of attack. The flowfield results of this problem using MacCormack's[4] method are given in Ref. 5 and so are not included here. Only the computational efficiency of Stetter's method is compared with that of MacCormack's in terms of the iterative time-steps and computing time required to achieve the steady-state solution.

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*Research Associate, ODU Research Foundation. Member AIAA.

†Research Leader, Aerothermodynamics Branch, Space Systems Division. Member AIAA.

Governing Equations and Solution Techniques

The development of the governing equations for the flow over a blunt axisymmetric body at zero angle of attack is given in Ref. 5. These equations in the body-oriented coordinate system are written as

$$\frac{\partial U'}{\partial t}+\frac{\partial M'}{\partial s}+\frac{\partial N'}{\partial n}+Q'=0 \tag{1}$$

where

$$U'=\lambda\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ \rho H-p\end{bmatrix}\qquad M'=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho uH\end{bmatrix}$$

$$N'=\lambda\begin{bmatrix}\rho v\\ \rho uv-\tau_l\\ \rho v^2+p\\ \rho vH-\dfrac{\mu}{\sigma Re}\dfrac{\partial h}{\partial n}-\mu\tau_l\end{bmatrix}$$

$$Q'=\frac{\lambda\sin\theta}{\beta}M'+\frac{\cos\theta}{\beta}N'+\begin{bmatrix}0\\ \kappa(\rho uv-\tau_l)-\dfrac{p\lambda\sin\theta}{\beta}\\ -\kappa(\rho u^2+p)-\dfrac{p\lambda\cos\theta}{\beta}\\ 0\end{bmatrix}$$

Here $\tau_l=\mu/Re(\partial u/\partial n-u\kappa/\lambda)$. In addition to the preceding conservation equations, an equation of state for the perfect gas is used along with Sutherland's viscosity law.

Two independent variable transformations are applied to Eq. (1). The first transformation maps the computational domain into a rectangular region in which both the shock and the body are made boundary mesh lines. This transformation is given by

$$\bar{y}=s,\ \ \bar{z}=1-n/\delta,\ \text{and}\ \ t=t \tag{2}$$

The second transformation further maps the computational region into another plane to allow higher resolution near the body surface. This is desirable to resolve the viscous region without too many mesh points in the normal direction. This transformation is given by

$$y=\bar{y},\ \ z=\ell n\left(\frac{\beta'+\bar{z}}{\beta'-\bar{z}}\right)\Big/\ell n\left(\frac{\beta'+1}{\beta'-1}\right),\ \text{and}\ \ t=t \tag{3}$$

Here β' is the stretching factor and is greater than 1. With the previous two transformations, Eq. (1) can be written as

$$\frac{\partial U}{\partial t}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z}+Q=0 \tag{4}$$

where U,M,N, and Q can be expressed suitably in terms of U',M',N', and Q'. More details can be found in Ref. 5. No-slip boundary conditions are used on the surface and the Rankine-Hugonoit relations are used to compute the flow conditions behind the shock. Equation (4) is solved by two finite-difference techniques; the first method is due to MacCormack[4] and the second method is due to Stetter.[3]

MacCormack's method: This two-step finite-difference method when applied to Eq. (4) can be written as

$$\overline{U^{k+1}_{n,m}} = U^k_{n,m} - \Delta t\{ (M^k_{n+1,m} - M^k_{n,m})/\Delta y + (N^k_{n,m+1} - N^k_{n,m})/\Delta z + Q^k_{n,m}\}$$

$$U^{k+1}_{n,m} = \tfrac{1}{2}[U^k_{n,m} + \overline{U^{k+1}_{n,m}} - \Delta t\{ (\overline{M^{k+1}_{n,m}} - \overline{M^{k+1}_{n-1,m}})/\Delta y + (\overline{N^{k+1}_{n,m}} - \overline{N^{k+1}_{n,m-1}})/\Delta z + \overline{Q^{k+1}_{n,m}}\}]$$

where Δy and Δz are the step sizes in the tangential and normal directions, respectively. Δt is the time-step size and is given by

$$\Delta t = R(\Delta t)_{\rm CFL} \tag{5}$$

A fourth-order damping term is added to the second step to damp the oscillations in the flow quantities. Although the method is simple to program and is second-order accurate in time and space, it often requires a large amount of computational time to obtain the steady-state solution. This is particularly true when a very fine mesh is employed. For the problem in which the transient solution is of no interest, a faster convergence can be achieved by marching in time at each mesh point according to its largest possible Courant-Friedrich-Lewy (CFL) time-step size. This procedure of marching in local time-step has been used in Ref. 6. A further reduction in the computational time is achieved by varying the stretching factor β' over the first few hundred time-steps and then keeping it constant for the rest of the calculations. This effectively means to start with a coarse mesh which is refined to the required mesh over the first few hundred time-steps. The net gain due to these modifications over the regular MacCormack method is discussed later.

A simple modification to MacCormack's method is proposed in Ref. 7 for faster convergence of the solution. This modification, called overrelaxation of MacCormack's scheme, has two parameters, $\bar{\omega}$ and ω. The regular MacCormack scheme is obtained by setting $\bar{\omega}=1$ and $\omega=0.5$. A faster convergence is expected for values of $\bar{\omega}$ and ω greater than 1.0 and 0.5, respectively. Although it is simple to implement in the existing computer programs, the gain in the present case was not found to be significant. The maximum values of $\bar{\omega}$ and ω, for which the program would work, were found to be 1.2 and 0.6, respectively. Even with these values, the value of R had to be reduced from 1.0 to 0.75. Thus, any gain due to overrelaxation is offset by the reduction in R.

Stetter's method: Equation (4) can be rewritten as

$$\partial U/\partial t = F = -(\partial M/\partial y + \partial N/\partial z + Q) \tag{6}$$

The three-step Stetter method, when applied to Eq. (6), is given by:

$$^{(0)}U^{k+1}_{n,m} = U^k_{n,m} + \Delta t F^k_{n,m}$$

$$^{(1)}U^{k+1}_{n,m} = U^k_{n,m} + \frac{\Delta t}{2}\left({}^{(0)}F^{k+1}_{n,m} + F^k_{n,m}\right)$$

$$^{(2)}U^{k+1}_{n,m} = U^k_{n,m} + \frac{\Delta t}{2}\left({}^{(1)}F^{k+1}_{n,m} + F^k_{n,m}\right)$$

$$U^{k+1}_{n,m} = \alpha\left({}^{(1)}U^{k+1}_{n,m}\right) + (1-\alpha)\left({}^{(2)}U^{k+1}_{n,m}\right) \tag{7}$$

The time-step is again given by Eq. (5).

The local value of the time-step is used at each mesh point. In all three steps, the spatial derivatives are obtained by central-difference relations. Also in this scheme the same fourth-order damping as used with MacCormack's scheme, is added in the final step of each iteration.

In the preceding difference equations, α is an adjustable parameter. As mentioned in Ref. 2, the values of α and R are determined numerically for a particular problem. It is found that for this problem $\alpha=0.1$ and $R=1.9$ most improved the convergence rate. The value of α could vary over a small range without significantly affecting the convergence. The results are obtained with constant stretching factor β' and also by varying β' over the first few hundred time-steps.

Discussion of Results

Solutions are obtained for the flow of air over a 45 deg half-angle hyperboloid at zero angle of attack. The freestream conditions are taken as $M_\infty = 12$, $T_\infty = 200$ K, $\rho_\infty = 1.2 * 10^{-2}$ Kg/m^3, and $Re = 3.07 \times 10^5$. The nose radius of the probe is taken as 0.1 m and the surface temperature is assumed to be 1200 K.

A mesh size of 11×101 is used. The mesh is equally spaced in the tangential direction with $\Delta y = 0.18$ and is refined in the normal direction by using a value of $\beta' = 1.05$. In the case when β' is variable, the value of β' is varied from 1.12 to 1.05 over the first few hundred time-steps and then held constant at 1.05 for the remaining time-steps. The computations are made on the CDC-STAR-100 computer for all the cases. The solution is assumed to be converged when the changes in the surface heating rates over 500 time-steps are less than 0.5% at all the mesh points. The results are presented in Table 1. It is seen from this table that the regular MacCormack scheme with global CFL time-step and constant β' required over 20,000 time-steps, whereas the local CFL time-step and constant β' required only 12,500 time-steps. Thus, the computational time is reduced to half just by the use of local time-step at each mesh point. A further reduction is obtained by varying β' from 1.12 to 1.05 over the first 3000 time-steps and then keeping it constant at 1.05. The solution is now converged in 11,000 time-steps.

Results are also given in Table 1 for Stetter's method. For this problem, $\alpha = 0.1$ and $R = 1.9$ gave the fastest convergence. It is seen that only 6500 time-steps are required for convergence with constant β' and local time-step at each mesh point. Since Stetter's method requires more work in each time-step, the reduction in the computational time is only about 25%, as compared to the corresponding case with MacCormack's method. However, when the problem includes real gas effects and radiative heat transfer, the calculations of these quantities require a very significant amount of computational time. For such problems, the total computational

Table 1 Computational results

	R $(=\Delta t/\Delta t_{\rm CFL})$	Number of time-steps	Computational time, s
MacCormack's method:			
a) Global minimum time-step and constant β'	1	Over 20,000	675 (for 20,000 time-steps)
b) local time-step and constant β'	1	12,500	420
c) local-time step and variable β'	1	11,000	375
Stetter's method:			
a) local time-step and constant β'	1.9	6500	325
b) local time-step and variable β'	1.9	6000	300

time will be substantially reduced by using Stetter's method which converges in a substantially lower number of time-steps. In addition, with Stetter's method a further reduction in computational time is obtained by using a variable β'. In this case, β' is varied from 1.12 to 1.05 over the first 1500 iterations and then held constant at 1.05. The converged solution is obtained in 6000 time-steps—a reduction of about 10% over the constant β' case.

The gain in using the local time-step and variable β' will increase with increasing Reynolds number, due to the requirement that at higher Reynolds numbers, a finer mesh near the body surface is required.

References

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